Name: ______ ______ Section: ______________

1. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where

$$A = \begin{bmatrix} 1 & -3 & -2 \\ -1 & 2 & 3 \\ 1 & -4 & -1 \end{bmatrix}, \quad \vec{\mathbf{u}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{\mathbf{b}} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{\mathbf{c}} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$$

(a) Compute $T(\vec{\mathbf{u}})$.

$T(\vec{u}) = A \cdot \vec{u}$

(b) Find all solutions to the equation $T(\vec{\mathbf{x}}) = \vec{\mathbf{b}}$

solve $A\vec{x} = \vec{b}$

(c) Is $\vec{\mathbf{c}}$ in the range of T? Justify your answer.

is $A\vec{x} = \vec{c}$ consistent?

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2. (No Computation) Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -3 & 2 \end{bmatrix}$$

(a) What is the domain of T? $\mathbb{R}^3$

$A\vec{x}$ defined $\Leftrightarrow$ $\vec{x}$ in $\mathbb{R}^3$

(b) What is the co-domain of T? $\mathbb{R}^2$

$A\vec{x}$ has 2 rows $\Rightarrow$ $A\vec{x}$ in $\mathbb{R}^2$

(c) Describe the Range of T as the span of a set of vectors.

outputs are obtained by scaling & adding col's of A

$\text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\} = \text{Range}(T).$

Complete answer

3. (No Computation) How many rows and columns must a matrix A have in order to define a mapping from $\mathbb{R}^5$ into $\mathbb{R}^7$ by the rule $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$?

inputs $= n$

outputs $= m$

A must be a 7×5 matrix

Complete answer

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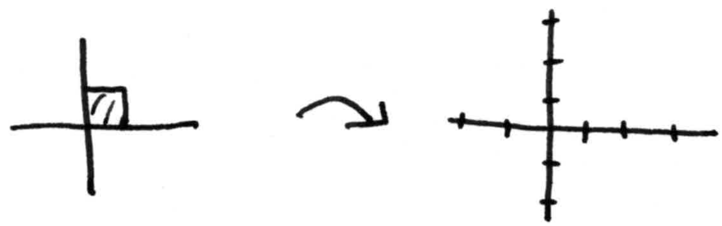
4. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$.

   Using two axes (one for inputs and one for outputs), show how T transforms the vertices $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Describe geometrically what the transformation T is doing (using words).



5. Find all the vectors $\vec{\mathbf{x}}$ that are mapped to $\vec{\mathbf{0}}$ by the transformation $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ where $A = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$,

Solve $T(\vec{x}) = \vec{0} \iff$ solve $A\vec{x} = \vec{0}$

$\iff$ reduce $\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \end{array}\right]$

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6. What is the definition of a Linear Transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$?

7. Let $\vec{\mathbf{x}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\vec{\mathbf{v}}_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, and $\vec{\mathbf{v}}_2 = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$, and let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation that maps $\vec{\mathbf{x}}$ to $x_1\vec{\mathbf{v}}_1 + x_2\vec{\mathbf{v}}_2$. Find a matrix A so that $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$ for every $\vec{\mathbf{x}}$.

$$T(\vec{x}) = x_1 \vec{v}_1 + x_2 \vec{v}_2$$

$$= x_1 \begin{bmatrix} 2 \\ -5 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

undo matrix multiplication.

$$=$$

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8. Find the standard matrix of the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that sends $\vec{\mathbf{e}}_1$ to $\vec{\mathbf{e}}_1 - 3\vec{\mathbf{e}}_2$ and leaves $\vec{\mathbf{e}}_2$ unchanged.

Recall that $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$T(e_1) =$

$T(e_2) =$

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8. Prove that $T(\vec{\mathbf{x}}) = 3\vec{\mathbf{x}}$ is a linear transformation.

Recall: T is linear $\Leftrightarrow$ ______________________ $P \leftrightarrow Q$

(Proof by computation)

Q

P

9. Prove that $T(\vec{\mathbf{x}}) = 3\vec{\mathbf{x}} + 1$ is not linear transformation.

Recall: T is linear $\Leftrightarrow$ ______________________ $P \leftrightarrow Q$

(Proof by computation)

$\neg Q$

$\neg P$

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10. Suppose that $\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_n$ span $\mathbb{R}^n$, and that T is a linear transformation with $T(\vec{\mathbf{v}}_1) = \vec{\mathbf{0}}, \ldots, T(\vec{\mathbf{v}}_n) = \vec{\mathbf{0}}$. Prove that $T(\vec{\mathbf{b}}) = \vec{\mathbf{0}}$ for *every* $\vec{\mathbf{b}} \in \mathbb{R}^n$.

proof by computation

introduce sutible notation

let $\vec{b}$ be any vector in $\mathbb{R}^n$

by definition of $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\} = \mathbb{R}^n$,

$\vec{b} = c_1 \cdot \vec{v}_1 + \cdots + c_n \vec{v}_n$

Compute

$T(\vec{b}) =$

$=$

$=$

$= \vec{0}$

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11. Let transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ be a linear transformation with
$T(\vec{\mathbf{e}}_1) = \begin{bmatrix}1\\3\end{bmatrix}$, $T(\vec{\mathbf{e}}_2) = \begin{bmatrix}-1\\3\end{bmatrix}$ and $T(\vec{\mathbf{e}}_3) = \begin{bmatrix}2\\4\end{bmatrix}$.

(a) Find the standard matrix A of T.

$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

(b) Determine if the transformation T is one-to-one.

T is one-to-one $\Leftrightarrow$ columns of A are independent

$\Leftrightarrow$ $A\vec{x} = \vec{0}$ has unique solution

(c) Determine if the transformation T is onto.

T is onto $\Leftrightarrow$ columns of A span $\mathbb{R}^2$

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12. Let transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with

$$T(\vec{\mathbf{e}}_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ T(\vec{\mathbf{e}}_2) = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} \text{ and } T(\vec{\mathbf{e}}_3) = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}.$$

(a) Find the standard matrix A of T.

(b) Determine if the transformation T is one-to-one.

(c) Determine if the transformation T is onto.

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14. Suppose that $T : \mathbb{R}^2 \to \mathbb{R}^3$ is a linear transformation with standard form matrix A.

    Prove that T is not onto. (Cite all relevant definitions and theorems by number).

proof by rephrasing

T is onto

$P \Leftrightarrow Q$ $\Leftrightarrow$ Theorem 12a

____

$Q \Leftrightarrow R$ $\Leftrightarrow$ Theorem 4

____

But

$\neg R$ $\boxed{}$

____

$\neg P$ So T is not onto

---

15. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^2$ is a linear transformation with standard form matrix A.

    Given this info ~~Can you prove that~~ do you know if T is onto? Justify your answer.

[redacted]

Hint: answer is "No, you cannot know if T is onto or not onto"

Why?

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Definitions

1. Define Span$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

$$= \left\{ \vec{b} \quad \text{s.t.} \quad \vec{b} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 \text{ for } c_1, c_2, c_3 \text{ in } \mathbb{R} \right\}$$

2. Define linear **Independence** of vectors $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$.

the set is independent

$\Longleftrightarrow$

$x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 = \vec{0}$ has ONLY the trivial solution

3. Define "T is a linear transformation"

$\Longleftrightarrow$

it satisfies $\left.\begin{array}{l} T(\vec{u}+\vec{v}) = T(\vec{u}) + T(\vec{v}) \\ T(c\cdot\vec{u}) = c\cdot T(\vec{v}) \end{array}\right\}$ for all $\vec{u}, \vec{v}$ & c

4. Define "T is one-to-one"

$\Longleftrightarrow$

$T(\vec{x}) = \vec{b}$ has at most one solution for each $\vec{b}$

5. Define "T is onto"

$\Longleftrightarrow$

$T(\vec{x}) = \vec{b}$ has at least one solution for each $\vec{b}$

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Theorems

Theorem 2 The reduced echelon form of a linear system has three possible cases

1. The system has 0 solutions if it contains [0 ⋯ 0 1 ■]
2. The system has 1 solutions if it has pivot in each variable column
3. The system has ∞-many solutions if it has variable column w/o pivot

Theorem 4: The columns of an $m \times n$ matrix A span $\mathbb{R}^m$

if and only if there is a pivot in each row

Theorem 5 If A is an $m \times n$ matrix, $\vec{\mathbf{u}}, \vec{\mathbf{v}} \in \mathbb{R}^n$ and $c \in \mathbb{R}$, *Then*

- $A(\vec{\mathbf{u}} + \vec{\mathbf{v}}) = A\vec{u} + A\vec{v}$ i.e. $T(\vec{x}) = A\vec{x}$ is linear.
- $A(c \cdot \vec{\mathbf{u}}) = c \cdot A\vec{u}$

Properties of Linear Transformations

- If T is linear, then $T(\vec{\mathbf{0}}) = \vec{0}$
- T is linear $\iff T(c \cdot \vec{\mathbf{u}} + d \cdot \vec{\mathbf{v}}) = c \cdot T(\vec{u}) + d \cdot T(\vec{v})$

Theorem 10 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear.

Then there is a unique $m \times n$ matrix A s.t. $T(\vec{\mathbf{x}}) = A\vec{\mathbf{x}}$.

In Fact, $A = \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}$ where $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

Theorem 12 Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear with standard matrix A. T

(a) T is onto $\iff$ columns of A span $\mathbb{R}^m$

(b) T is one-to-one $\iff$ columns of A are linearly independent